

# Complexity computation for compact 3-manifolds via crystallizations and Heegaard diagrams

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## Abstract

The idea of computing Matveev complexity by using Heegaard decompositions has been recently developed by two different approaches: the first one for closed 3-manifolds via crystallization theory, yielding the notion of *Gem-Matveev complexity*; the other one for compact orientable 3-manifolds via generalized Heegaard diagrams, yielding the notion of *modified Heegaard complexity*. In this paper we extend to the non-orientable case the definition of modified Heegaard complexity and prove that for closed 3-manifolds Gem-Matveev complexity and modified Heegaard complexity coincide. Hence, they turn out to be useful different tools to compute the same upper bound for Matveev complexity.

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## 1. Introduction

In 1990 S. Matveev proposed in [22] to attack the problem of studying systematically the whole set  $\mathcal{M}$  of compact 3-manifolds by choosing a suitable notion of *complexity*, i.e. a non-negative function which filters  $\mathcal{M}$  and is able to “*measure how complicated a combinatorial description of the manifold must be*”. If the filtration has the properties of finiteness (only a finite number of closed irreducible 3-manifolds have a fixed complexity) and additivity with respect to connected sum (the complexity of the connected sum is the sum of the complexities of the summands), then it allows a concrete catalogation of the elements of  $\mathcal{M}$ , via the chosen combinatorial tool. In the same paper, Matveev introduced a notion of complexity with the required properties, based on the theory of simple spines ([21] and [29]).

We recall that a polyhedron  $P$  embedded into a compact connected 3-manifold  $M$  is called a *spine* of  $M$  if  $M$  (or  $M$  minus an open 3-ball if  $M$  is closed) collapses to  $P$ . Moreover, a spine

$S$  is said to be *almost simple* if the link of each point  $x \in S$  can be embedded into  $K_4$ , which is the topological realization of the complete graph with four vertices. A *true vertex* of an almost simple spine  $S$  is a point  $x \in S$  whose link is homeomorphic to  $K_4$ .

The (*Matveev*) *complexity*  $c(M)$  of  $M$  is defined as the minimum number of true vertices among all almost simple spines of  $M$ . The 3-sphere, the real projective space, the lens space  $L(3, 1)$  and the spherical bundles  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$  have complexity zero by definition. Apart from these special cases, for a closed prime manifold  $M$ , the complexity  $c(M)$  turns out to be the minimum number of tetrahedra needed to obtain  $M$  via face paring of them ([22, Proposition 2], together with the related Remark).

During the last two decades, various authors produced tables of closed 3-manifolds for increasing values of complexity, by simply generating all triangulations (resp. spines) with a given number of tetrahedra (resp. true vertices) and classifying topologically the associated manifolds. The obtained results concerning the orientable (resp. non-orientable) case may be found in [22], [28], [20], [26], [24], [25] and in the Web page <http://www.matlas.math.csu.ru/> (resp. in [6], [1], [9], [2] and [7, Appendix]).

In general, the computation of the complexity of a given manifold is a difficult problem (see [17] and [18] for recent results). So, two-sided estimates of complexity become important, especially when dealing with infinite families of manifolds (see, for example, [23], [27], [30]). By [23, Theorem 2.6.2], a lower bound for the complexity of a given manifold can be obtained from its first homology group. Moreover, a lower bound for hyperbolic manifolds can be obtained via volume arguments (see [23], [27], [30]). Upper bounds are easier to find, since any pseudo-triangulation (or any spine) of  $M$  obviously yields an upper bound for  $c(M)$ .

The idea of computing Matveev complexity by using Heegaard decompositions is already suggested in the foundational paper [22] by Matveev: from any Heegaard diagram  $H = (S, v, w)$  of  $M$ , we can construct an almost simple spine of  $M$  whose true vertices are the intersection points of the curves of the two systems  $v$  and  $w$ , with the exception of those which lie on the boundary of a region of  $S - \{v \cup w\}$ . In fact, the spine can be obtained by adding to the surface  $S$  the meridian disks corresponding to the systems of curves and by removing the 2-disk corresponding to an arbitrary region of  $S - \{v \cup w\}$ .

Starting from this idea, two different approaches to Matveev complexity computation have been recently developed. The first one, introduced in 2004 for closed 3-manifolds, is based on crystallization theory; it has led to the notion of *Gem-Matveev complexity*, *GM-complexity* for short (see [9], together with subsequent papers [10] and [11], or Section 3 of the present paper for a brief account). Later, in 2010, the *modified Heegaard complexity* (*HM-complexity*) of a compact orientable 3-manifold has been defined via generalized Heegaard diagrams (see [12]). Both invariants have been proved to be upper bounds for the Matveev complexity.

From the practical view-point, both *GM-complexity* and *HM-complexity* have allowed to obtain estimations of complexity for interesting classes of manifolds. In [10] *GM-complexity* has produced significant improvements in order to estimate Matveev complexity for two-fold branched coverings of  $\mathbb{S}^3$ , three-fold simple branched coverings of  $\mathbb{S}^3$  and 3-manifolds obtained by Dehn surgery on framed links in  $\mathbb{S}^3$ . On the other hand, estimations for  $n$ -fold cyclic coverings of  $\mathbb{S}^3$  branched over 2-bridge knots and links, torus knots and theta graphs, as well as for a wide class of Seifert manifolds which generalize Neuwirth manifolds have been obtained through *HM-complexity* in [12]. Note also that, in [8], *GM-complexity* has allowed us to complete the classification of all non-orientable closed 3-manifold up to complexity 6 (see [1] and [2]).

The aim of the present paper is to extend the definition of modified Heegaard complexity to the

non-orientable case (Section 2), and to prove that for each closed 3-manifold Gem-Matveev complexity and modified Heegaard complexity coincide (Proposition 6). Furthermore, experimental results concerning 3-manifolds admitting a crystallization with “few” vertices suggest equality between Matveev complexity and this upper bound, directly computable via two apparently different methods for representing 3-manifolds (Conjecture 7).

## 2. Modified Heegaard complexity

The notion of *modified Heegaard complexity* for compact orientable 3-manifolds (either with or without boundary) has been introduced in [12], where a comparison with Matveev complexity has been discussed. In this section we extend that notion to the non-orientable case. In order to do that, some preliminary definitions are required.

Let  $\Sigma_g$  be either the closed, connected orientable surface of genus  $g$  (with  $g \geq 0$ ) or the closed, connected non-orientable surface of genus  $2g$  (with  $g \geq 1$ ). So  $\Sigma_g$  is the boundary of a handlebody  $\mathbb{Y}_g$  of genus  $g$ ,  $\mathbb{Y}_g$  being the orientable (resp. non orientable) 3-manifold obtained from the 3-ball  $\mathbb{D}^3$  by adding  $g$  orientable 1-handles (resp.  $g$  1-handles, at least one of which is non-orientable).

A *system of curves* on  $\Sigma_g$  is a (possibly empty) set of simple closed orientation-preserving<sup>1</sup> curves  $\mathcal{C} = \{\gamma_1, \dots, \gamma_k\}$  on  $\Sigma_g$  such that  $\gamma_i \cap \gamma_j = \emptyset$ , for  $1 \leq i \neq j \leq k$ . Moreover, we denote with  $V(\mathcal{C})$  the set of connected components of the surface obtained by cutting  $\Sigma_g$  along the curves of  $\mathcal{C}$ . The system  $\mathcal{C}$  is said to be *proper* if all elements of  $V(\mathcal{C})$  have genus zero, and *reduced* if either  $|V(\mathcal{C})| = 1$  or no element of  $V(\mathcal{C})$  has genus 0. Thus,  $\mathcal{C}$  is: (i) proper and reduced if and only if  $V(\mathcal{C})$  consists of one element of genus 0; (ii) non-proper and reduced if and only if all elements of  $V(\mathcal{C})$  are of genus  $> 0$ ; (iii) proper and non-reduced if and only if  $V(\mathcal{C})$  has more than one element and all of them are of genus 0; (iv) non-proper and non-reduced if and only if  $V(\mathcal{C})$  has at least one element of genus 0 and at least one element of genus  $> 0$ . Note that a proper reduced system of curves on  $\Sigma_g$  contains exactly  $g$  curves.

We denote by  $G(\mathcal{C})$  the graph which is dual to the one determined by  $\mathcal{C}$  on  $\Sigma_g$ . Thus, vertices of  $G(\mathcal{C})$  correspond to elements of  $V(\mathcal{C})$  and edges correspond to curves of  $\mathcal{C}$ . Note that loops and multiple edges may arise in  $G(\mathcal{C})$ .

A *compression body*  $K_g$  of genus  $g$  is a 3-manifold with boundary obtained from  $\Sigma_g \times [0, 1]$  by attaching a finite set of 2-handles  $Y_1, \dots, Y_k$  along a system of curves (called *attaching circles*) on  $\Sigma_g \times \{0\}$  and filling in with balls all the spherical boundary components of the resulting manifold, except  $\Sigma_g \times \{1\}$  when  $g = 0$ . Moreover,  $\partial_+ K_g = \Sigma_g \times \{1\}$  is called the *positive* boundary of  $K_g$ , while  $\partial_- K_g = \partial K_g - \partial_+ K_g$  is called the *negative* boundary of  $K_g$ . Notice that a compression body is a handlebody if and only if  $\partial_- K_g = \emptyset$ , i.e., the system of the attaching circles on  $\Sigma_g \times \{0\}$  is proper. Obviously homeomorphic compression bodies can be obtained via (infinitely many) non isotopic systems of attaching circles.

If a system of attaching circles  $\mathcal{C}$  is not reduced, then it contains at least one reduced subsystem of curves determining the same compression body  $K_g$ . Indeed, let  $V^+(\mathcal{C})$  be the set of vertices of  $G(\mathcal{C})$  corresponding to the components with genus greater than zero, and  $\mathcal{A}(\mathcal{C})$  be the set consisting of all the graphs  $T_i$  such that:

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<sup>1</sup>This means that each curve  $\gamma_i$  has an annular regular neighborhood, as it always happens if  $\Sigma_g$  is an orientable surface.

- $T_i$  is a subgraph of  $G(\mathcal{C})$ ;
- if  $V^+(\mathcal{C}) = \emptyset$  then  $T_i$  is a maximal tree in  $G(\mathcal{C})$ ;
- if  $V^+(\mathcal{C}) \neq \emptyset$  then  $T_i$  contains all the vertices of  $G(\mathcal{C})$  and each component of  $T_i$  is a tree containing exactly one vertex of  $V^+(\mathcal{C})$ .

Then, for any  $T_i \in \mathcal{A}(\mathcal{C})$ , the system of curves obtained by removing from  $\mathcal{C}$  the curves corresponding to the edges of  $T_i$  is reduced and determines the same compression body. Note that this operation corresponds to removing complementary 2- and 3-handles. Moreover, if  $\partial_- K_g$  is orientable (resp. non-orientable) and has  $h$  boundary components with genus  $g_j$  (resp.  $2g_j$ ),  $1 \leq j \leq h$ , then<sup>2</sup>

$$|E(T_i)| = |\mathcal{C}| - g - \max\{0, h - 1\} + \sum_{j=1}^h g_j$$

for each  $T_i \in \mathcal{A}(\mathcal{C})$ , where  $E(T_i)$  denotes the edge set of  $T_i$ .

Let  $M$  be a compact, connected 3-manifold without spherical boundary components. A *Heegaard surface* for  $M$  is a surface  $\Sigma_g$  embedded in  $M$  such that  $M - \Sigma_g$  consists of two components whose closures  $K'$  and  $K''$  are (homeomorphic to) a genus  $g$  handlebody and a genus  $g$  compression body, respectively.

The triple  $(\Sigma_g, K', K'')$  is called a *Heegaard splitting* of genus  $g$  of  $M$ . It is a well known fact that each compact connected 3-manifold without spherical boundary components admits a Heegaard splitting.

**Remark 1** By Proposition 2.1.5 of [25], the complexity of a manifold is not affected by puncturing it. So, in order to compute complexity, there is no loss of generality to assume that the manifold has no spherical boundary components.

On the other hand, a triple  $H = (\Sigma_g, \mathcal{C}', \mathcal{C}'')$ , where  $\mathcal{C}'$  and  $\mathcal{C}''$  are two systems of curves on  $\Sigma_g$ , such that they intersect transversally and  $\mathcal{C}'$  is proper, uniquely determines a 3-manifold  $M_H$  corresponding to the Heegaard splitting  $(\Sigma_g, K', K'')$ , where  $K'$  and  $K''$  are respectively the handlebody and the compression body whose attaching circles correspond to the curves in the two systems. Such a triple is called a *generalized Heegaard diagram* for  $M_H$ .

In the case of closed 3-manifolds, both systems of curves of a generalized Heegaard diagram  $H$  are obviously proper; if they are also reduced,  $H$  is simply a Heegaard diagram in the classical sense (see [15]).

For each generalized Heegaard diagram  $H = (\Sigma_g, \mathcal{C}', \mathcal{C}'')$ , we denote by  $\Delta(H)$  the graph embedded in  $\Sigma_g$  defined by the curves of  $\mathcal{C}' \cup \mathcal{C}''$ , and by  $\mathcal{R}(H)$  the set of regions of  $\Sigma_g - \Delta(H)$ . Note that  $\Delta(H)$  may have connected components which are circles. All vertices not belonging in these components are 4-valent and they are called *singular* vertices. A diagram  $H$  is called *reduced Heegaard diagram* if both the systems of curves are reduced. If  $H$  is non-reduced, then we denote by  $\text{Rd}(H)$  the set of reduced Heegaard diagrams obtained from  $H$  by reducing the two systems of curves.

The modified complexity of a reduced Heegaard diagram  $H'$  is

$$\tilde{c}(H') = c(H') - \max\{n(R) \mid R \in \mathcal{R}(H')\},$$

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<sup>2</sup>The formula corrects a misprint contained in [12].

where  $c(H')$  is the number of singular vertices of  $\Delta(H')$  and  $n(R)$  denotes the number of singular vertices contained in the region  $R$ ; while the modified complexity of a (non-reduced) generalized Heegaard diagram  $H$  is

$$\tilde{c}(H) = \min \{ \tilde{c}(H') \mid H' \in \text{Rd}(H) \}.$$

We define the *modified Heegaard complexity* of a compact connected 3-manifold  $M$  as

$$c_{HM}(M) = \min \{ \tilde{c}(H) \mid H \in \mathcal{H}(M) \},$$

where  $\mathcal{H}(M)$  is the set of all generalized Heegaard diagrams of  $M$ .

The significance of modified Heegaard complexity consists in its relation with Matveev complexity  $c(M)$ :

**Proposition 1** *If  $M$  is a compact connected 3-manifold, then*

$$c(M) \leq c_{HM}(M).$$

**Proof.** The result has been proved in [12] for compact orientable manifolds, but the proof works exactly in the same way also for non-orientable ones. ■

### 3. Crystallizations and GM-complexity

The present section is devoted to briefly review some basic notions of the representation theory of PL-manifolds by crystallizations; in particular, we focus on definitions and results (due to [9], [10] and [11]) concerning the possibility of obtaining an upper bound for Matveev complexity of a closed 3-manifold  $M$  by means of the edge-coloured graphs representing  $M$ .

For general PL-topology and elementary notions about graphs and embeddings, we refer to [16] and [31] respectively.

Crystallization theory is a representation tool for general piecewise linear (PL) compact manifolds, without assumptions about dimension, connectedness, orientability or boundary properties (see the survey papers [13], [3] and [5]). However, since this paper concerns only 3-manifolds, we will restrict definitions and results to dimension 3, although they mostly hold for the general case ( $n \geq 1$ ); moreover, from now on all manifolds will be assumed to be closed and connected.

Given a pseudocomplex  $K$ , triangulating a 3-manifold  $M$ , a *coloration* on  $K$  is a labelling of its vertices by  $\Delta_3 = \{0, 1, 2, 3\}$ , which is injective on each simplex of  $K$ . The dual 1-skeleton of  $K$  is a (multi)graph  $\Gamma = (V(\Gamma), E(\Gamma))$  embedded in  $|K| = M$ ; we can define a map  $\gamma : E(\Gamma) \rightarrow \Delta_3$  in the following way:  $\gamma(e) = c$  iff the vertices of the face dual to  $e$  are coloured by  $\Delta_3 - \{c\}$ . The map  $\gamma$  - which is injective on each pair of adjacent edges of the graph - is called an *edge-coloration* on  $\Gamma$ , while the pair  $(\Gamma, \gamma)$  is called a *4-coloured graph representing  $M$*  or simply a *gem* (where “gem” stands for *graph encoded manifold*: see [19]). In order to avoid long notations, in the following we will often omit the edge-coloration when it is not necessary, and we will simply write  $\Gamma$  instead of  $(\Gamma, \gamma)$ .

It is easy to see that any 3-manifold  $M$  has a gem inducing it: just take the barycentric subdivision  $H'$  of any pseudocomplex  $H$  triangulating  $M$ , label any vertex of  $H'$  with the dimension of the open simplex containing it in  $H$ , and construct the associated 4-coloured graph as described

above. Conversely, starting from  $\Gamma$ , we can always reconstruct  $K(\Gamma) = K$  and hence the manifold  $M$  (see [13] and [3] for more details).

Given  $i, j \in \Delta_3$ ,  $i \neq j$ , we denote by  $(\Gamma_{i,j}, \gamma_{i,j})$  the 2-coloured graph obtained from  $\Gamma$  by deleting all edges which are not  $i$ - or  $j$ -coloured; hence,  $\Gamma_{i,j} = (V(\Gamma), \gamma^{-1}(\{i, j\}))$  and  $\gamma_{i,j} = \gamma|_{\gamma^{-1}(\{i, j\})}$ . The connected components of  $\Gamma_{i,j}$  will be called  $\{i, j\}$ -*residues* of  $\Gamma$  and their number will be denoted by  $g_{i,j}$ . As a consequence of the definition, a bijection is established between the set of  $\{i, j\}$ -residues of  $\Gamma$  and the set of 1-simplices of  $K(\Gamma)$  whose endpoints are labelled by  $\Delta_3 - \{i, j\}$ . Moreover, for each  $c \in \Delta_3$ , the connected components of the 3-coloured graph  $\Gamma_{\hat{c}}$  obtained from  $\Gamma$  by deleting all  $c$ -coloured edges are in bijective correspondence with the  $c$ -coloured vertices of  $K(\Gamma)$ ; their number will be denoted by  $g_{\hat{c}}$ . We will call  $\Gamma$  *contracted* iff  $\Gamma_{\hat{c}}$  is connected for each  $c \in \Delta_3$ , i.e. iff  $K(\Gamma)$  has exactly four vertices.

A contracted 4-coloured graph representing a 3-manifold  $M$  is called a *crystallization* of  $M$ . It is well-known that every 3-manifold admits a crystallization (see [13], together with its references). Any crystallization (or more generally any gem)  $\Gamma$  of  $M$  encodes in a combinatorial way the topological properties of  $M$ . For example, it is very easy to check that  $M$  is orientable iff  $\Gamma$  is bipartite.

Relations among crystallization theory and other classical representation methods for PL manifolds have been widely analyzed (see [3, Sections 3, 6, 7]). In particular, for our purposes, it is useful to recall how crystallizations and Heegaard diagrams are strongly correlated.

A cellular embedding  $\iota$  of a 4-coloured graph  $\Gamma$  into a surface is said to be *regular* if there exists a cyclic permutation  $\varepsilon$  of  $\Delta_3$  such that the regions of  $\iota$  are bounded by the images of  $\{\varepsilon_j, \varepsilon_{j+1}\}$ -residues of  $\Gamma$  ( $j \in \mathbb{Z}_4$ ). If  $\Gamma$  is a bipartite (resp. non-bipartite) crystallization of a 3-manifold  $M$ , for each pair  $\alpha, \beta \in \Delta_3$ , let us set  $\{\alpha', \beta'\} = \Delta_3 - \{\alpha, \beta\}$  and let  $F_{\alpha, \beta}$  be the orientable (resp. non orientable) surface of genus  $g_{\alpha, \beta} - 1 = g_{\alpha', \beta'} - 1$ , obtained from  $\Gamma$  by attaching a 2-cell to each  $\{i, j\}$ -residue such that  $\{i, j\} \neq \{\alpha, \beta\}$  and  $\{i, j\} \neq \{\alpha', \beta'\}$ . This construction proves the existence of a regular embedding  $\iota_{\alpha, \beta} : \Gamma \rightarrow F_{\alpha, \beta}$ . Moreover, if  $\mathcal{D}$  (resp.  $\mathcal{D}'$ ) is an arbitrarily chosen  $\{\alpha, \beta\}$ -residue (resp.  $\{\alpha', \beta'\}$ -residue) of  $\Gamma$ , the triple  $\mathcal{H}_{\alpha, \beta, \mathcal{D}, \mathcal{D}'} = (F_{\alpha, \beta}, \mathbf{x}, \mathbf{y})$ , where  $\mathbf{x}$  (resp.  $\mathbf{y}$ ) is the set of the images of all  $\{\alpha, \beta\}$ -residues (resp.  $\{\alpha', \beta'\}$ -residues) of  $\Gamma$ , except  $\mathcal{D}$  (resp.  $\mathcal{D}'$ ), is a Heegaard diagram of  $M$ . Conversely, given a Heegaard diagram  $\mathcal{H} = (F, \mathbf{x}, \mathbf{y})$  of  $M$  and  $\alpha, \beta \in \Delta_3$ , there exists a construction which, starting from  $\mathcal{H}$  yields a crystallization  $\Gamma$  of  $M$  such that  $\mathcal{H} = \mathcal{H}_{\alpha, \beta, \mathcal{D}, \mathcal{D}'}$  for a suitable choice of  $\mathcal{D}$  and  $\mathcal{D}'$  in  $\Gamma$  (see [14]).

Now, let us denote by  $\mathcal{R}_{\mathcal{D}, \mathcal{D}'}$  the set of regions of  $F_{\alpha, \beta} - (\mathbf{x} \cup \mathbf{y}) = F_{\alpha, \beta} - \iota_{\alpha, \beta}((\Gamma_{\alpha, \beta} - \mathcal{D}) \cup (\Gamma_{\alpha', \beta'} - \mathcal{D}'))$ .

**Definition 1.** Let  $M$  be a closed 3-manifold, and let  $(\Gamma, \gamma)$  be a crystallization of  $M$ . With the above notations, the *Gem-Matveev complexity* (or *GM-complexity*, for short) of  $\Gamma$  is defined as the non-negative integer

$$c_{GM}(\Gamma) = \min\{\#V(\Gamma) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) \mid \alpha, \beta \in \Delta_3, \mathcal{D} \subset \Gamma_{\alpha, \beta}, \mathcal{D}' \subset \Gamma_{\alpha', \beta'}, \Xi \in \mathcal{R}_{\mathcal{D}, \mathcal{D}'}\},$$

while the (*non-minimal*) *GM-complexity* of  $M$  is defined as the minimum value of *GM-complexity*,

where the minimum is taken over all<sup>3</sup> crystallizations of  $M$  :

$$c'_{GM}(M) = \min\{c_{GM}(\Gamma) \mid (\Gamma, \gamma) \text{ crystallization of } M\}.$$

The following key result, due to [9], justifies the choice of terminology:

**Proposition 2** *For every closed 3-manifold  $M$ , Gem-Matveev complexity gives an upper bound for Matveev complexity of  $M$ :*

$$c(M) \leq c'_{GM}(M).$$

Unfortunately, the edge-coloured graphs which are obtained, by suitable constructions, from different representations of manifolds are mostly non-contracted. Therefore, the above definitions need slight modifications in order to be useful for the general case of a non-contracted gem  $(\Gamma, \gamma)$  of  $M$ .

For each pair  $\alpha, \beta \in \Delta_3$ , let  $K_{\alpha, \beta}$  be the 1-dimensional subcomplex of  $K(\Gamma)$  generated by the  $\{\alpha, \beta\}$ -coloured vertices. Moreover, let  $\mathbf{D} = \{\mathcal{D}_1, \dots, \mathcal{D}_{g_{\hat{\alpha}}+g_{\hat{\beta}}-1}\}$  (resp.  $\mathbf{D}' = \{\mathcal{D}'_1, \dots, \mathcal{D}'_{g_{\hat{\alpha}'}+g_{\hat{\beta}'}-1}\}$ ) be a collection of  $\{\alpha, \beta\}$ -coloured (resp.  $\{\alpha', \beta'\}$ -coloured) cycles of  $(\Gamma, \gamma)$  corresponding to a maximal tree of  $K_{\alpha, \beta}$  (resp.  $K_{\alpha', \beta'}$ ); we denote by  $\mathcal{R}_{\mathbf{D}, \mathbf{D}'}$  the set of regions of  $F_{\alpha, \beta} - \iota_{\alpha, \beta}((\Gamma_{\alpha, \beta} - \cup_{i=1, \dots, g_{\hat{\alpha}}+g_{\hat{\beta}}-1} \mathcal{D}_i) \cup (\Gamma_{\alpha', \beta'} - \cup_{j=1, \dots, g_{\hat{\alpha}'}+g_{\hat{\beta}'}-1} \mathcal{D}'_j))$ ,  $\iota_{\alpha, \beta} : \Gamma \rightarrow F_{\alpha, \beta}$  being a regular embedding of  $\Gamma$  into the (orientable or non-orientable, according to the bipartition of  $\Gamma$ ) closed surface of genus  $g_{\alpha, \beta} - g_{\hat{\alpha}} - g_{\hat{\beta}} + 1$ .

**Definition 2.** Let  $M$  be a closed 3-manifold and let  $(\Gamma, \gamma)$  be an edge-coloured graph representing  $M$ . With the above notations, the *GM-complexity* of  $\Gamma$  is defined as the non-negative integer

$$c_{GM}(\Gamma) = \min\{\#V(\Gamma) - \# \left[ \left( \bigcup_{\mathcal{D}_i \in \mathbf{D}} V(\mathcal{D}_i) \right) \cup \left( \bigcup_{\mathcal{D}'_j \in \mathbf{D}'} V(\mathcal{D}'_j) \right) \cup V(\Xi) \right] \mid \alpha, \beta \in \Delta_3, \right. \\ \left. \mathbf{D} \subset \Gamma_{\alpha, \beta}, \mathbf{D}' \subset \Gamma_{\alpha', \beta'}, \Xi \in \mathcal{R}_{\mathbf{D}, \mathbf{D}'} \}.$$

Note that if  $\Gamma$  is contracted, the maximal tree of  $K_{\alpha, \beta}$  (resp.  $K_{\alpha', \beta'}$ ) consists of one edge, therefore  $\mathbf{D}$  (resp.  $\mathbf{D}'$ ) contains exactly one  $\{\alpha, \beta\}$ -coloured (resp.  $\{\alpha', \beta'\}$ -coloured) cycle. Hence the above definition agrees with Definition 1 in the case of crystallizations.

**Definition 3.** Let  $M$  be a closed 3-manifold. The *extended GM-complexity* of  $M$  is defined as the minimum value of *GM-complexity*, where the minimum is taken over all edge-coloured graphs representing  $M$  (without assumptions about contractedness):

$$\tilde{c}_{GM}(M) = \min\{c_{GM}(\Gamma) \mid |K(\Gamma)| = M\}.$$

The following result - due to [10] - allows to consider non-minimal *GM-complexity* and extended *GM-complexity* as “improvements” of Gem-Matveev one, in order to estimate Matveev complexity.

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<sup>3</sup>Note that the original paper [9] introduces also the notion of *Gem-Matveev complexity* (or *GM-complexity* for short) of  $M$  - denoted by  $c_{GM}(M)$  - as the minimum value of GM-complexity, where the minimum is taken only over crystallizations of  $M$  which are minimal with respect to the order of the graph. Obviously,  $c'_{GM}(M) \leq c_{GM}(M)$  for every  $M$ .

**Proposition 3** *For every closed 3-manifold  $M$ , the following chain of inequalities holds:*

$$c(M) \leq \tilde{c}_{GM}(M) \leq c'_{GM}(M).$$

## 4. Proof of the main result

In this section we prove the announced equality between the two (apparently) different approaches to Matveev complexity described in the previous sections.

**Lemma 4** *For every closed 3-manifold  $M$ , the following inequality holds:*

$$c'_{GM}(M) \leq c_{HM}(M).$$

**Proof.** First of all, we can suppose  $c_{HM}(M) \neq 0$ , since for each closed 3-manifold with Matveev complexity zero  $c'_{GM}(M) = 0$  holds (see [9] and [11]).

Let now  $\bar{H} = (\Sigma_g, \mathcal{C}', \mathcal{C}'')$  be a generalized Heegaard diagram of  $M$ , such that  $c_{HM}(\bar{H}) = c_{HM}(M) = \bar{c}$ . By definition, there exists a reduced Heegaard diagram  $\bar{H}'$  of  $M$  ( $\bar{H}' \in \text{Rd}(\bar{H})$ ), with  $c_{HM}(\bar{H}') = \bar{c}$ ; let  $\bar{R} \in \mathcal{R}(\bar{H}')$  be the region of  $\Delta(\bar{H}')$  such that  $c_{HM}(\bar{H}') = c(\bar{H}') - n(\bar{R})$  ( $n(\bar{R})$  being the number of singular vertices contained in  $\bar{R}$ ).

We are going to apply to  $\bar{H}'$  the construction described in [14, Lemma 4]. Note that the hypothesis  $c_{HM}(\bar{H}') = c_{HM}(M)$  directly implies that  $\bar{H}'$  satisfies condition (a) of [14, Lemma 4]. Moreover, condition (b) of the cited Lemma may also be assumed without affecting  $c_{HM}(\bar{H}')$ .

Let us first suppose that the reduced Heegaard diagram  $\bar{H}'$  is such that the graph  $\Gamma'$  imbedded in  $\Lambda_{2g}$  consisting of all the curves of  $\mathcal{C}''$  and two copies of each curve of  $\mathcal{C}'$  (see [14, p. 476] for details) is connected. In this case,  $\bar{H}'$  trivially satisfies also condition (c) of [14, Lemma 4], unless there is no intersection between the curves of  $\mathcal{C}'$  and  $\mathcal{C}''$  (i.e. unless  $\bar{H}'$  contradicts the hypothesis  $c_{HM}(M) \neq 0$ ). As a consequence, it is possible to construct a crystallization  $\bar{\Gamma}$  of  $M$ , such that one of its associated Heegaard diagrams is exactly  $\bar{H}'$ : this means that a  $\{0, 2\}$ -residue  $\mathcal{D}$  (resp. a  $\{1, 3\}$ -residue  $\mathcal{D}'$ ) of  $\bar{\Gamma}$  exists so that  $\bar{H}' = \mathcal{H}_{0,2,\mathcal{D},\mathcal{D}'} = (F_{0,2}, \mathbf{x}, \mathbf{y})$ , where  $F_{0,2}$  is the surface of genus  $g_{0,2} - 1$  into which  $\bar{\Gamma}$  regularly embeds via  $\iota_{0,2}$  and  $\mathbf{x}$  (resp.  $\mathbf{y}$ ) is the set of the images in  $\iota_{0,2}$  of all  $\{0, 2\}$ -residues (resp.  $\{1, 3\}$ -residues) of  $\bar{\Gamma}$  but  $\mathcal{D}$  (resp.  $\mathcal{D}'$ ) (see Section 2). It is now easy to check that  $\bar{R} \in \mathcal{R}(\bar{H}')$  corresponds to a region  $\Xi \in \mathcal{R}_{\mathcal{D},\mathcal{D}'}$ , where  $\mathcal{R}_{\mathcal{D},\mathcal{D}'}$  denotes the set of regions of  $F_{0,2} - (\mathbf{x} \cup \mathbf{y})$ . Hence, by definition,  $c_{GM}(\bar{\Gamma}) \leq \#V(\bar{\Gamma}) - \#(V(\mathcal{D}) \cup V(\mathcal{D}') \cup V(\Xi)) = c(\bar{H}') - n(\bar{R}) = \bar{c}$ . In this case, the thesis  $c'_{GM}(M) \leq c_{HM}(M)$  directly follows.

Let us now assume that the reduced Heegaard diagram  $\bar{H}'$  is such that the graph  $\Gamma'$  imbedded in  $\Lambda_{2g}$  is not connected. If  $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_h$  ( $h \geq 2$ ) denote its connected components, then the reduced Heegaard diagram  $\bar{H}'$  splits into  $h$  Heegaard diagrams  $\bar{H}'_1, \bar{H}'_2, \dots, \bar{H}'_h$ , where  $\bar{H}'_i = (\Sigma_{g_i}, \mathcal{C}'_i, \mathcal{C}''_i)$  is such that  $\sum_{i=1}^h g_i = g$ ,  $\Sigma_g = \#_{i=1}^h \Sigma_{g_i}$  and  $\cup_{i=1}^h \mathcal{C}'_i = \mathcal{C}'$  (resp.  $\cup_{i=1}^h \mathcal{C}''_i = \mathcal{C}''$ ). Note that  $\bar{R}$  is the only region of  $\Delta(\bar{H}')$  obtained by “fusing” the regions  $\bar{R}_1, \bar{R}_2, \dots, \bar{R}_h$  ( $\bar{R}_i$  being a suitable region of  $\Delta(\bar{H}'_i)$  with  $n(\bar{R}_i) \neq 0$  singular vertices, and  $\sum_{i=1}^h n(\bar{R}_i) = n(\bar{R})$ ).<sup>4</sup> In fact, if this

<sup>4</sup>Roughly speaking, we can say that  $\bar{R}$  is the “external” region of the embedding of the Heegaard diagram  $\bar{H}'$  in  $\Lambda_{2g}$ , and that  $\bar{R}_i$  is the “external” region of the embedding of the Heegaard diagram  $\bar{H}'_i$  in  $\Lambda_{2g_i}$ , for each  $i = 1, \dots, h$ .



is not the case, it is easy to check that a new Heegaard diagram  $\bar{H}''$  of  $M$  with this property exists, with  $c_{HM}(\bar{H}'') < c_{HM}(\bar{H}')$ . Moreover,  $c_{HM}(\bar{H}'_i) = c(\bar{H}'_i) - n(\bar{R}_i)$  trivially holds, together with  $c_{HM}(\bar{H}'_i) = c_{HM}(M_i)$ ,  $M_i$  ( $i = 1, \dots, h$ ) being the 3-manifold represented by the Heegaard diagram  $\bar{H}'_i$ , so that  $M = \#_{i=1}^h M_i$ . Hence,  $c_{HM}(M) = \sum_{i=1}^h c_{HM}(M_i)$ .

On the other hand, if  $\bar{\Gamma}$  (resp.  $\bar{\Gamma}^{(i)}$ ,  $\forall i = 1, \dots, h$ ) is the crystallization of  $M$  (resp. of  $M_i$ ) obtained from  $\bar{H}'$  (resp.  $\bar{H}'_i$ ) by the procedure of [14, Lemma 4], then  $\bar{\Gamma}$  may be trivially obtained by graph connected sum (see [13]) from  $\bar{\Gamma}^{(1)}, \bar{\Gamma}^{(2)}, \dots, \bar{\Gamma}^{(h)}$ .

Now, since  $\bar{H}'_i$  is such that the graph  $\Gamma'_i$  is connected, the above discussion ensures  $c_{GM}(\bar{\Gamma}^{(i)}) \leq c_{HM}(\bar{H}'_i)$ , for each  $i = 1, \dots, h$ .

Finally,  $c_{GM}(\bar{\Gamma}) \leq \sum_{i=1}^h c_{GM}(\bar{\Gamma}^{(i)})$  trivially holds by construction. The thesis now directly follows:  $c'_{GM}(M) \leq c_{GM}(\bar{\Gamma}) \leq \sum_{i=1}^h c_{GM}(\bar{\Gamma}^{(i)}) = \sum_{i=1}^h c'_{GM}(M_i) = c_{HM}(M)$ . ■

**Lemma 5** *For every closed 3-manifold  $M$ , the following inequality holds:*

$$c_{HM}(M) \leq \tilde{c}_{GM}(M).$$

**Proof.** Let  $\Gamma$  be a gem of  $M$  such that  $c_{GM}(\Gamma) = \tilde{c}_{GM}(M)$  and let  $\alpha, \beta \in \Delta_3$  be such that the minimal  $GM$ -complexity of  $\Gamma$  is obtained by means of the regular embedding associated to  $\alpha$  and  $\beta$ . Moreover, if  $K_{\alpha, \beta}$  (resp.  $K_{\alpha', \beta'}$ ) is the 1-dimensional subcomplex of  $K(\Gamma)$  generated by the  $\{\alpha, \beta\}$ -coloured (resp.  $\{\alpha', \beta'\}$ -coloured) vertices, there exist a maximal tree of  $K_{\alpha, \beta}$  (resp.  $K_{\alpha', \beta'}$ ) and an element  $\Xi \in \mathcal{R}_{\mathbf{D}, \mathbf{D}'}$  ( $\mathbf{D}$  and  $\mathbf{D}'$  being the collections of  $\{\alpha, \beta\}$ - and  $\{\alpha', \beta'\}$ -coloured cycles corresponding to the maximal trees of  $K_{\alpha', \beta'}$  and  $K_{\alpha, \beta}$  respectively) such that

$$c_{GM}(\Gamma) = \#V(\Gamma) - \#(V(\mathbf{D}) \cup V(\mathbf{D}') \cup V(\Xi)),$$

where  $V(\mathbf{D})$  (resp.  $V(\mathbf{D}')$ ) denotes the set of the vertices of all the cycles in  $\mathbf{D}$  (resp.  $\mathbf{D}'$ ).

Let  $\bar{K}$  be the largest 2-dimensional subcomplex of the first barycentric subdivision of  $K(\Gamma)$  disjoint from the first barycentric subdivisions of  $K_{\alpha, \beta}$  and  $K_{\alpha', \beta'}$ . The surface  $F$ , triangulated by  $\bar{K}$ , splits  $K(\Gamma)$  into two polyhedra  $\mathcal{A}_{\alpha, \beta}$  and  $\mathcal{A}_{\alpha', \beta'}$  whose intersection is exactly  $F$ . Moreover  $F = F_{\alpha, \beta} = F_{\alpha', \beta'}$  (where - according to the previous section -  $F_{\alpha, \beta}$  and  $F_{\alpha', \beta'}$  are the surfaces into which  $\Gamma$  regularly embeds via  $\iota_{\alpha, \beta}$  and  $\iota_{\alpha', \beta'}$  respectively).

Both  $\mathcal{A}_{\alpha, \beta}$  and  $\mathcal{A}_{\alpha', \beta'}$  are compression bodies. In fact, we can think  $\mathcal{A}_{\alpha, \beta}$  (resp.  $\mathcal{A}_{\alpha', \beta'}$ ) as constructed by considering a collar of  $F$  and by adding on  $F \times \{1\}$  the 2-handles whose attaching spheres are all the  $\{\alpha', \beta'\}$ -coloured (resp.  $\{\alpha, \beta\}$ -coloured) cycles of  $\Gamma$ , except those of  $\mathbf{D}'$  (resp.  $\mathbf{D}$ ).

Therefore, we consider the generalized Heegaard diagram of  $M$  given by  $H = (F, \mathcal{C}', \mathcal{C}'')$ , where  $\mathcal{C}'$  and  $\mathcal{C}''$  are the systems of curves on  $F$  defined by the attaching cycles described above.

Actually,  $\mathcal{A}_{\alpha, \beta}$  (resp.  $\mathcal{A}_{\alpha', \beta'}$ ) is a handlebody of genus  $g_{\alpha', \beta'} - \#\mathbf{D}'$  (resp.  $g_{\alpha, \beta} - \#\mathbf{D}$ ), since it collapses to the graph  $K_{\alpha, \beta}$  (resp.  $K_{\alpha', \beta'}$ ).

As a consequence  $g(F) = g_{\alpha', \beta'} - \#\mathbf{D}' = g_{\alpha, \beta} - \#\mathbf{D}$  and both  $\mathcal{C}'$  and  $\mathcal{C}''$  are proper and reduced. Since the number of singular vertices of  $H$  is exactly  $\#V(\Gamma) - \#(V(\mathbf{D}) \cup V(\mathbf{D}'))$  and  $\Xi$  obviously corresponds to a region of  $H$  having the maximal vertex number, we have  $c_{HM}(H) = c_{GM}(\Gamma)$ , hence  $c_{HM}(M) \leq \tilde{c}_{GM}(M)$ . ■

**Remark 2** The proof of Lemma 5 shows that *any gem  $\Gamma$  of a closed 3-manifold  $M$  induces three generalized Heegaard diagrams for  $M$ , one for each choice of a pair of different colours  $\alpha, \beta \in \Delta_3$ .*

Moreover, the sets of all  $\{\alpha, \beta\}$ - and all  $\{\alpha', \beta'\}$ -cycles of  $\Gamma$  are two proper systems of curves on the surface  $F_{\alpha, \beta}$ , which are always non-reduced. In the case of a crystallization, a reduced diagram may be simply obtained by removing an arbitrary curve from both systems of curves (i.e. the sets  $\mathbf{D}$  and  $\mathbf{D}'$  are the smallest possible, each consisting of only one element).

As a direct consequence of Lemma 4 and Lemma 5, together with Proposition 3, the equality among the three notions follows:

**Proposition 6** *For every closed 3-manifold  $M$ ,*

$$c_{HM}(M) = c'_{GM}(M) = \tilde{c}_{GM}(M).$$

Hence, both modified Heegaard complexity and non-minimal  $GM$ -complexity and extended  $GM$ -complexity turn out to be different tools to compute the same upper bound for Matveev complexity.

Actually, by experimental results of [8] and [4], this upper bound is proved to be sharp (i.e.:  $c(M) = c_{HM}(M) = c'_{GM}(M) = \tilde{c}_{GM}(M)$ ) for the thirty-eight (resp. sixteen) closed connected prime orientable (resp. non-orientable) 3-manifolds admitting a coloured triangulation with at most 26 (resp. 30) tetrahedra. As far as we know, there is no example where the strict inequality holds.

Hence, we formulate the following:

**Conjecture 7** *For every closed connected 3-manifold  $M$ ,*

$$c(M) = c_{HM}(M) = c'_{GM}(M) = \tilde{c}_{GM}(M).$$

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